2025 VCE Mathematical Methods 1 external assessment report

Areas for improvement

Mathematical notation

Mathematical notation is a precise language and students should be familiar with command terms as outlined in the examination specifications and study design.

Where an equation or rule is requested, the response must clearly present the required rule in full, rather than providing an incomplete statement such as $c = \ldots$ Students should also use the names of functions as given in the question.

If a question gives a rule for a function as $y =$ and requires that $\frac{dy}{dx}$ be found, it is important to name the derivative $\frac{dy}{dx}$; $y =$ is not acceptable. An integral or integration statement must be accompanied by a 'dx' as in $\int f(x)dx$ and $\log_e$ has the e written as a subscript.

Correctly using brackets eliminates the potential for ambiguity or errors in calculations and ensures that the order of operations is clear. In particular, students need to consider the use of brackets when enacting the product and quotient rules for differentiation; often the x^2 and $-\sin x$ appeared as a difference of two terms rather than a product. Brackets around an interval indicate that all values between the endpoints are included, with curved and square brackets having specific meaning.

Presentation

Students should read the questions carefully and prepare their responses in the required format.

Students should ensure that their final answer is obvious as the intended response for marking. Where multiple, conflicting answers are given, full marks cannot be awarded. Students should ensure that any workings completed separately are clearly and accurately incorporated into their final solution, if required.

Drawing graphs

When drawing a graph, students are encouraged to use pencil for curve sketching, but all components should be distinct enough to appear clearly. Very light or dashed lines, such as those used for asymptotes, may not be visible for the assessors and should be drawn dark enough to be visible. When adding details to graphs, students should ensure their handwriting is clear.

In addition, students should also use the provided grid to ensure that their graph line passes through points on the correct Cartesian path. In the case of a cosine curve, sinusoidal properties (in particular, symmetry) must be clearly demonstrated, with the graph exhibiting mirror-image properties about the vertical axis of symmetry. When labelling graphs, coordinate pairs need to be labelled with curved brackets. Students are also encouraged to pay attention to domain restrictions.

Specific information

This report provides sample answers, or an indication of what answers may have included. Unless otherwise stated, these are not intended to be exemplary or complete responses.

The statistics in this report may be subject to rounding resulting in a total more or less than 100 per cent.

Question 1a

Marks	0	1	Average
%	12	88	0.9

$$\frac{dy}{dx} = 2x\cos(x) + x^2 \cdot \left(-\sin(x)\right)$$
$$= 2x\cos(x) - x^2\sin(x) = x\left(2\cos(x) - x\sin(x)\right)$$

This question was well attempted and required students to use the product rule to find the derivative. Many students did not tidy up the negative signs in their answer and left their answer as $2x\cos(x) + -x^2\sin(x)$ or $2x\cos(x) + x^2 - \sin(x)$. Some students did not use brackets around terms, which meant that these presentations had the potential to be misinterpreted. Although not required, some students decided to factorise their answer and take x out as a common factor.

Question 1b

Marks	0	1	2	Average
%	18	21	61	1.5

$$f(x) = 6\sqrt{x+1} + 5 = 6(x+1)^{\frac{1}{2}} + 5$$

$$f'(x) = 6 \cdot \frac{1}{2} \cdot (x+1)^{-\frac{1}{2}} = 3(x+1)^{-\frac{1}{2}} = \frac{3}{\sqrt{x+1}}$$

$$f'(8) = \frac{3}{\sqrt{8+1}}$$
$$= 1$$

This question was well attempted and required students to use the chain rule to find the derivative, then evaluate the derivative at $x = 8$. Some students did not correctly execute the chain rule and omitted the numerator. Some students did not correctly identify the initial power of the $(x+1)$ term as $\frac{1}{2}$. Some students incorrectly thought $\sqrt{9} = \pm 3$, leading to an incorrect answer of ± 1. A correct answer must emerge from correct working. Some students proceeded further to find the equation of the tangent line instead of only finding the gradient of the tangent at $x = 8$, as required. Students are reminded to carefully read the question and answer what is required.

Question 2

Marks	0	1	2	Average
%	22	38	40	1.2

$$g'(x) = \frac{1}{2x+3} \text{ where } x > -\frac{3}{2} \text{ and } g(1) = 0$$

$$g(x) = \int \frac{1}{2x+3}\,dx = \frac{1}{2}\log_e(2x+3) + c$$

$$g(1) = 0 = \frac{1}{2}\log_e(5) + c, \quad c = -\frac{1}{2}\log_e(5)$$

$$\therefore \quad g(x) = \begin{cases} \frac{1}{2}\log_e(2x+3) - \frac{1}{2}\log_e(5) \\ \frac{1}{2}\log_e\left(\frac{2x+3}{5}\right) & \text{equivalent answer} \\ \log_e\left(\sqrt{\frac{2x+3}{5}}\right) & \text{equivalent answer} \end{cases}$$

Any equivalent form.

This question required students to antidifferentiate $g(x)$ to obtain a $\log_e$ term plus the constant of integration, c. Using the condition $g(1)=0$ allowed the value of c to be calculated. Some students chose to take out a factor of $\frac{1}{2}$ to obtain an equivalent antiderivative $\frac{1}{2}\log_e(x+\frac{3}{2})+c$. Upon substituting $g(1)=0$ this yielded the same answer for $g(x)$. Some students did not recognise that the integral would involve a logarithmic function. A common error was writing $\ln(2x+3)+c$ as the first step, without accounting for the necessary factor of $\frac{1}{2}$ in front of $\ln(2x+3)$. A common incorrect answer was $\log_e\left(\frac{2x+3}{5}\right)$.

Question 3a

Marks	0	1	Average
%	19	81	0.8

$[-1, 3]$ or $-1 \le f(x) \le 3$ or between −1 and 3 inclusive

This question was well answered. The most common errors were to write the interval with curved brackets $(-1, 3)$ or with incorrectly signed values as $[1, 3]$. Some students incorrectly wrote $[3, -1]$. It is important to note that incorrectly stating the range of values in terms of x (as in $-1 \le x \le 3$) is not acceptable notation.

Question 3b

Marks	0	1	2	3	Average
%	18	13	18	51	2.0

$$\cos(2x) = -\frac{1}{2}$$
$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Or use a general solution equation and find particular values for $0 \le x \le 2\pi$.

$$2x = \pm\frac{2\pi}{3} + 2k\pi,$$
$$x = \pm\frac{\pi}{3} + k\pi,$$

$$k = 0, \quad x = \cancel{-\frac{\pi}{3}}, \quad \frac{\pi}{3}$$

$$k = 1, \quad x = \frac{2\pi}{3}, \quad \frac{4\pi}{3};$$

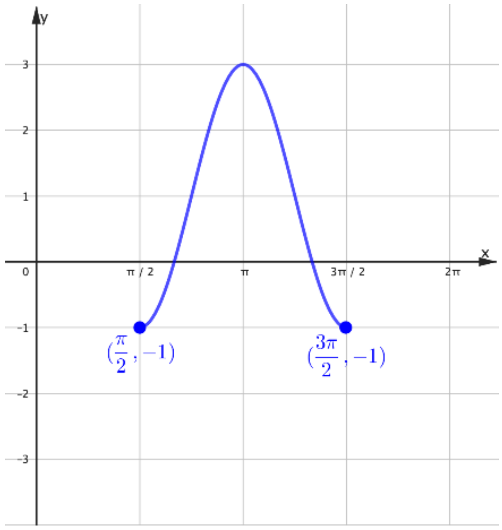
$$k = 2, \quad x = \frac{5\pi}{3}, \quad \cancel{\frac{7\pi}{3}}$$

$$\therefore \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

This question required particular solutions to be found to a trigonometric equation within the domain $0 \le x \le 2\pi$. Some students could not identify the correct angle or quadrant for the initial angle. Students are reminded that the exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for values of θ between 0 and $\frac{\pi}{2}$ are expected key knowledge for the study, as specified in the study design. Some students only gave two of the solutions, not taking into account the period of the function. Some students gave a general solution to the equation without indicating the particular solutions.

Question 3c

Marks	0	1	2	Average
%	34	32	33	1.0



Most students presented cosine graphs that were drawn over the correct domain and range. Students generally included details and labels as required and produced smooth graph lines that displayed appropriate sinusoidal behaviour. Students are encouraged to pay attention to the symmetry of the curve and to use the grid lines to assist with accurately positioning the curve. Common errors included labelling the endpoints incorrectly as $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, sketching a graph over the range $[-1, 2]$, extending the graph beyond the domain $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, or not passing the graph through the maximum point of $(\pi, 3)$. Some students incorrectly sketched an inverted version of the graph. Some students positioned their x-intercepts incorrectly and/or asymmetrically and some students drew graphs that looked more like parabolas.

Question 4a

Marks	0	1	2	Average
%	26	18	56	1.3

The sum of all probabilities is 1:

$$\frac{4}{k}+\frac{2k}{75}+\frac{k}{75}+\frac{2}{k}=1$$

$$\Rightarrow \frac{6}{k}+\frac{3k}{75}=1$$

$$\Rightarrow 3k^2+450=75k$$

$$\Rightarrow 3k^2-75k+450=0$$

$$\Rightarrow k^2-25k+150=0$$

$$\Rightarrow (k-10)(k-15)=0$$

$$\therefore \quad k=10 \quad \text{or} \quad k=15$$

Or use the quadratic formula:

$$k=\frac{25\pm\sqrt{625-600}}{2}=\frac{25\pm 5}{2}=15,10$$

This question was a 'show that' question. As such, each line of working needed to demonstrate a clear, logical and explicit progression, leading to the result provided in the question stem. In particular, the probabilities needed to be added together and equated to 1, a quadratic equation formed and correctly solved. It was not sufficient to verify the solutions of $k=10$ or $k=15$ by substitution. Some students did not form the correct quadratic equation. Some students incorrectly used the formula for $E(X)$ instead of using the fact that the probabilities must sum to 1.

Question 4b.i

Marks	0	1	Average
%	25	75	0.8

$$\Pr(X>1)=\Pr(X=2)+\Pr(X=3)$$

$$=\frac{1}{5}\cdot\frac{3}{3}+\frac{2}{15}$$

$$=\frac{25}{75}=\frac{5}{15}=\frac{1}{3}$$

This question was well attempted. Common errors included incorrectly using $\Pr(X>1)=1-\Pr(X=0)$ or $\Pr(X>1)=\Pr(X=1)+\Pr(X=2)+\Pr(X=3)$, both of which led to the incorrect answer $\frac{11}{15}$. Some students incorrectly cancelled $\frac{5}{15}$ to $\frac{1}{5}$.

Question 4b.ii

Marks	0	1	Average
%	29	71	0.7

$$\begin{aligned}\mathrm{E}(X) &= 0\times\frac{4}{15}+1\times\frac{30}{75}+2\times\frac{15}{75}+3\times\frac{2}{15} \\ &= \frac{30}{75}+\frac{30}{75}+\frac{30}{75}=\frac{6}{15}+\frac{6}{15}+\frac{6}{15} \\ &= \frac{90}{75}=\frac{18}{15}=\frac{6}{5}\quad\left(\text{or }\ 1.2\ \text{ or }\ 1\frac{1}{5}\right)\end{aligned}$$

This question was well attempted. Some students made arithmetic errors when working with the fractions.

Question 5a

Marks	0	1	2	Average
%	14	25	61	1.5

$$e^{2x}-8e^{x}+7=0$$
$$\left(e^{x}-1\right)\left(e^{x}-7\right)=0$$

$$e^{x}=1 \quad\text{or}\quad e^{x}=7$$
$$x=0 \quad\text{or}\quad x=\log_e(7)$$

Or

$$\begin{aligned}&let\ u=e^{x}\\ &u^2-8u+7=0\\ &(u-7)(u-1)=0\\ &e^{x}=7\ \text{ or }\ e^{x}=1\\ &x=\log_e 7\ \text{ or }\ x=0\end{aligned}$$

This question required the solution of a quadratic equation involving exponential terms. Most students recognised the quadratic nature of the question and were able to set up, factorise and solve correctly. Some students incorrectly discarded the solution $x=\ln(1)$. Some students did not observe $\log_e(1)=0$.

Question 5b

Marks	0	1	2	Average
%	42	16	43	1.0

Method 1:

Find turning point x-value

$$g'(x) = 2e^{2x} - 8e^{x} = 0$$
$$2e^{x}\left(e^{x} - 4\right) = 0$$
$$\text{As } e^{x} > 0, \quad e^{x} = 4 \text{ only}$$
$$\therefore \quad x = \log_e(4)$$

So $a = \log_e(4) = 2\log_e(2)$

Method 2:

As e^{x} is an increasing function over R

$(u-7)(u-1) = 0$

Axis of symmetry $u = 4$

$$e^{x} = 4$$
$$\therefore \quad x = \log_e(4)$$

So $a = \log_e(4) = 2\log_e(2)$

This question involved finding the location of the turning point, either by using calculus or symmetry. Although students were generally able to differentiate and set the derivative equal to zero, some students made arithmetic errors when factorising, which led to an incorrect result of $e^{x} = 8$ and hence an incorrect value $a = \ln(8)$. Some students chose to use a substitution and let $a = e^{x}$, leading to a solution of $a = 4$. Rather than recognising that this meant $e^{x} = 4$, they then incorrectly concluded that $a = 4$ was the answer to the upper bound of the interval. Students are advised to be careful when introducing variables.

Question 6a

Marks	0	1	Average
%	27	73	0.8

$$\text{var}\left(X\right) = np(1-p) = 6 \times \frac{1}{4} \times \frac{3}{4}$$
$$= \frac{18}{16} = \frac{9}{8} \qquad \left(= 1\frac{1}{8} = 1.125\right)$$

This question was well answered. Common mistakes included finding the standard deviation instead of the variance and incorrect multiplication of the fractions.

Question 6b

Marks	0	1	2	Average
%	31	35	34	1.0

$$\begin{aligned}\Pr(X \geq 5) &= \Pr(X=5)+\Pr(X=6)\\ &= \binom{6}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^1+\binom{6}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^0\\ &= 6\times\frac{1\times 3}{4^6}+\frac{1}{4^6}\\ &= \frac{18+1}{4^6}\\ &= \frac{19}{2^{12}}\end{aligned}$$

Many students correctly stated the binomial expansion sum with appropriate probability values and powers. The answer was required to be stated in a particular form, but some students were not able to reduce 4096 down to the prime factorisation of 2^{12}. Instead, many students gave the answer as $\frac{19}{4^6}$. Many students calculated only one term, $\Pr(X=5)$, instead of evaluating $\Pr(X=5)+\Pr(X=6)$.

Question 7a

Marks	0	1	Average
%	10	90	0.9

$$\begin{aligned}f(5) &= 125-25-16(5)-20\\ &= 125-25-80-20\\ &= 125-125\\ &= 0\end{aligned}$$

$\therefore \quad x=5$ is a solution to $f(x)=0$

This question was well answered. Some students chose to use a factor theorem approach to ‘show that’ $x=5$ was a solution and, although not necessary, this approach was appropriate.

Question 7b

Marks	0	1	2	Average
%	17	8	76	1.6

Method 1: long division

$$\begin{array}{r l} & \quad\quad x^2+4x+4 \\ (x-5)\overline{)} & x^3-x^2-16x-20 \\ & -\left(x^3-5x^2\right) \\ \hline & \quad 4x^2-16x \\ & -\left(4x^2-20x\right) \\ \hline & \quad\quad 4x-20 \\ & -\left(4x-20\right) \\ \hline & \quad\quad\quad 0 \end{array}$$

$$\begin{aligned} \therefore \quad f(x) &= (x-5)\left(x^2+4x+4\right) \\ &= (x+2)^2(x-5) \end{aligned}$$

Method 2: equate coefficients

$(x^2+2dx+d^2)(x-5) = f(x)$

$= x^3-5x^2+2dx^2-10dx+d^2x-5d^2$

$= x^3+(2d-5)x^2+(d^2-10d)x-5d^2$

$\therefore 2d-5=-1$ and $d^2-10d=-16$ and $5d^2=20$

$\therefore d=2$

$f(x)=(x+2)^2(x-5)$

Method 3: synthetic division

$$\begin{array}{c|cccc} 5 & 1 & -1 & -16 & -20 \\ & & 5 & 20 & 20 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$\Rightarrow \dfrac{f(x)}{x-5} = x^2+4x+4$

$$\begin{aligned} \therefore \quad f(x) &= (x-5)\left(x^2+4x+4\right) \\ &= (x+2)^2(x-5) \end{aligned}$$

Method 4: factorising by grouping

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 4x^2 - 20x + 4x - 20 \\ &= x^2(x-5) + 4x(x-5) + 4(x-5) \\ &= (x^2+4x+4)(x-5) \\ &= (x+2)^2(x-5) \end{aligned}$$

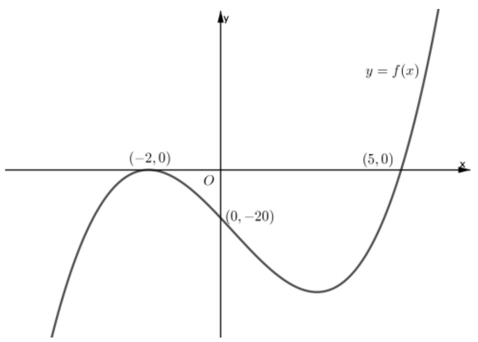
Method 5: find missing terms

$$\begin{aligned} x^3 - x^2 - 16x - 20 &= (x-5)(x^2+4x+4) \\ &= (x-5)(x+2)^2 \end{aligned}$$

This question was well answered, with students using a variety of valid methods such as long division, synthetic division, and expanding and equating coefficients to factorise $f(x)$. However, some students gave the correct answer without showing any working to support the answer. Students are reminded that for any question worth more than one mark, working must be shown in order to be awarded full marks.

Question 7c

Marks	0	1	Average
%	24	76	0.8



This question was well answered. A common error was omitting the negative sign in the intercept coordinates, giving $(2,0)$ and $(0,20)$ instead of the correct $(-2,0)$ and $(0,-20)$.

Question 7d.i

Marks	0	1	Average
%	56	44	0.5

$$\begin{aligned} y &= f(x)g(x) \\ &= (x+2)^2(x-5)(x+2) \\ &= (x+2)^3(x-5) \end{aligned}$$

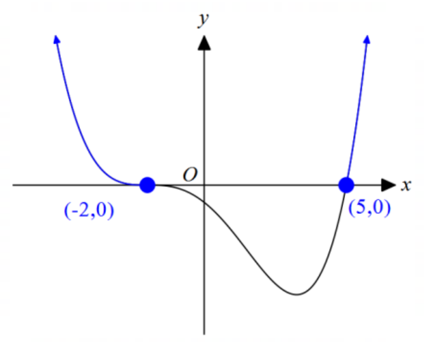
The triple factor $(x+2)^3$ indicates $(-2, 0)$ is the stationary point of inflection required.

Some students gave the equation of the product curve $(x+2)^3(x-5)$ as their answer, rather than the coordinate of the stationary point of inflection. Some students chose to use the expanded form of $f(x)$ and then expand the product instead of using the result from **part b** to express the product in factorised form. This approach often made the question unnecessarily difficult and prevented some students from reaching the correct answer.

Question 7d.ii

Marks	0	1	Average
%	73	27	0.3

Graph of $y=(x+2)^3(x-5)$



To have $f(x)g(x) \geq 0$, then

$x \leq -2$ or $x \geq 5$

Alternatively in interval notation

$(-\infty, -2] \cup [5, \infty)$ or $R \setminus (-2, 5)$

This question was not well answered. Although not required, consideration of the shape of the graph of the equation $y = f(x)g(x)$ could be used to assist with determining the interval required. Common incorrect answers included $(-\infty, 2] \cup [5, \infty)$ (missing a negative sign in front of 2), $[-2, 5]$, $[5, \infty) \cup \{-2\}$ and, less frequently, $(-\infty, 2] \cap [5, \infty)$.

Question 8a

Marks	0	1	2	3	Average
%	30	23	23	24	1.4

Method 1:

$$\begin{aligned}\Pr(X>k) &= \int_k^{\frac{4}{3}} \frac{3}{8}(4-3x)\,dx \\ &= -\frac{3}{8}\int_k^{\frac{4}{3}}(3x-4)\,dx \\ &= -\frac{3}{8}\cdot\frac{1}{2(3)}\left[(3x-4)^2\right]_k^{\frac{4}{3}} \\ &= -\frac{1}{16}\left[(4-4)^2-(3k-4)^2\right] \\ &= \frac{(3k-4)^2}{16}\end{aligned}$$

$$\begin{aligned}\frac{(3k-4)^2}{16}=\frac{9}{16} &\Rightarrow (3k-4)^2=9 \\ &\Rightarrow 3k-4=\pm 3 \\ &\Rightarrow 3k=7 \quad \text{or} \quad 3k=1\end{aligned}$$

$\cancel{k=\frac{7}{3}}$ rejected as it is outside $\left[0,\frac{4}{3}\right]$

$$\therefore \quad k=\frac{1}{3} \text{ only}$$

Method 2:

$$1-\Pr(0\le X\le k)=\frac{9}{16}$$

$$\Pr(0\le X\le k)=\int_0^k \frac{3}{8}(4-3x)\,dx=\frac{7}{16}$$

$$\int_0^k (4-3x)\,dx=\frac{7}{6}$$

$$\left[4x-\frac{3x^2}{2}\right]_0^k=\frac{7}{6}$$

$$4k-\frac{3k^2}{2}=\frac{7}{6}$$

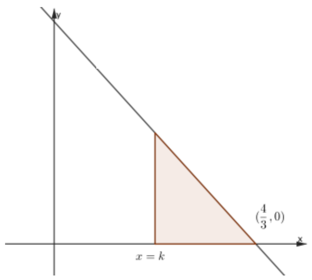
$$24k-9k^2=7$$

$$9k^2-24k+7=0$$

$$(3k-7)(3k-1)=0$$

$$\cancel{k=\frac{7}{3}} \quad \text{or} \quad k=\frac{1}{3}$$

Method 3:



Could use triangle area under the straight line.

$$\frac{1}{2}\times\frac{3}{8}(4-3k)\times\left(\frac{4}{3}-k\right)=\frac{9}{16}$$

$$3(4-3k)\times\left(\frac{4-3k}{3}\right)=9$$

$$(4-3k)^2=9$$

$$(3k-4)^2=9 \Rightarrow 3k-4=\pm 3$$

$$\text{thus } \cancel{k=\frac{7}{3}}, \quad k=\frac{1}{3}$$

Or

$$0=9k^2-24k+7$$

$$0=(3k-1)(3k-7)$$

$$k=\frac{1}{3} \text{ or } k=\frac{7}{3}$$

$$\therefore k=\frac{1}{3}$$

Or

$$k=\frac{8\pm\sqrt{64-4\times 3\times\frac{7}{3}}}{6}$$

$$k=\frac{8\pm\sqrt{36}}{6}$$

$$k=\frac{2}{6} \text{ or } k=\frac{14}{6}$$

$$\therefore k=\frac{1}{3}$$

Students could either set up the integral $\int_{k}^{\frac{4}{3}} \frac{3}{8}(4-3x)\,dx = \frac{9}{16}$, the integral $\int_{0}^{k} \frac{3}{8}(4-3x)\,dx = \frac{7}{16}$ or form an area equation using a triangle or trapezium. Most students were able to form a definite integral and integrate correctly to obtain a quadratic equation. Some students who chose to integrate the $(4-3x)$ term as a bracketed term (as in method 1), raised the power to 2 but then divided by 2 instead of 6. Some students were unable to form the correct quadratic equation or solve it correctly. The quadratic expression was readily factorised by inspection, but a large proportion of students used the quadratic formula or other techniques such as splitting the middle term and grouping. Some students did not convert fractional coefficients into integers before solving; for example, using $\frac{3}{2}k^2 - 4k + \frac{7}{6} = 0$ instead of the simpler $9k^2 - 24k + 7 = 0$.

Students who obtained the two possible solutions were mostly aware of rejecting $\frac{7}{3}$.

Question 8b

Marks	0	1	2	Average
%	51	22	27	0.8

Method 1:

$$\begin{aligned}\int_0^{\frac{4}{3}} h(x)\,dx &= \int_0^{\frac{4}{3}} \left(mf(x)+n\right)dx \\ &= m\underbrace{\int_0^{\frac{4}{3}} f(x)\,dx}_{=1 \text{ as it's a p.d.f}} + \int_0^{\frac{4}{3}} n\,dx \\ &= m + \frac{4}{3}n \quad = \quad \frac{3m+4n}{3} \quad = \quad \frac{1}{3}(3m+4n)\end{aligned}$$

or other equivalent forms

Method 2:

$$\int_0^{\frac{4}{3}} \left(mf(x)+n\right)dx = m\int_0^{\frac{4}{3}} f(x)\,dx + \int_0^{\frac{4}{3}} n\,dx$$

$$= \frac{3m}{8}\left[4x-\frac{3x^2}{2}\right]_0^{\frac{4}{3}} + [nx]_0^{\frac{4}{3}}$$

$$= \frac{3m}{8}\left(\frac{16}{3}-\frac{8}{3}\right)+\frac{4}{3}n$$

$$= \begin{cases} m+\frac{4}{3}n \\ = \dfrac{3m+4n}{3} \\ = \dfrac{1}{3}(3m+4n) \end{cases}$$

or other equivalent forms

This question was not responded to well. Most students correctly rewrote $h(x)$ as $m\left(\frac{3}{8}(4-3x)\right)+n$.

However, many students incorrectly proceeded to factor out *m* from the entire integral without noticing that this was not algebraically valid. The students who recognised that the total probability is equal to 1 and applied it, were generally successful.

Question 9a

Marks	0	1	2	3	Average
%	47	25	10	19	1.0

Method 1: quartic

$$\frac{w^2}{(x-1)^2} = (x-w)^2, \text{ with } w=-3$$

$$\frac{9}{(x-1)^2} = (x+3)^2 \Rightarrow (x+3)^2(x-1)^2 = 9$$

$$\Rightarrow (x^2-2x+1)(x^2+6x+9)=9$$

$$\Rightarrow x^4+4x^3-2x^2-12x+9=9$$

$$\Rightarrow x^4+4x^3-2x^2-12x=0$$

$$\Rightarrow x\left(x^3+4x^2-2x-12\right)=0$$

$$\Rightarrow x(x+2)(x^2+2x-6)=0$$

can use CTS or QF

$$\therefore x=0,\ x=-2,\ x=\frac{-2\pm\sqrt{4+24}}{2}$$

$$x=0,\ x=-2,\ x=-1\pm\sqrt{7}$$

Method 2: DOPS

$$\frac{w^2}{(x-1)^2}=(x-w)^2\text{, with } w=-3$$

$$\frac{9}{(x-1)^2}=(x+3)^2 \Rightarrow (x+3)^2(x-1)^2=9$$
$$\Rightarrow (x+3)^2(x-1)^2-3^2=0$$
$$\Rightarrow \left[(x+3)(x-1)-3\right]\left[(x+3)(x-1)+3\right]=0$$
$$\Rightarrow \left(x^2+2x-6\right)\left(x^2+2x\right)=0$$

$$\begin{aligned}\text{case 1: } x^2+2x-6&=0\\ x^2+2x+1&=7\\ (x+1)^2&=7\\ \Rightarrow x&=-1\pm\sqrt{7}\end{aligned}$$

$$\begin{aligned}\text{case 2: } x^2+2x&=0,\\ x(x+2)&=0\\ \Rightarrow x=0,\quad x&=-2\end{aligned}$$

All four solutions are valid – none of them equals to 1.

$x \in R\setminus\{1\}$ is satisfied.

Method 3:

$\dfrac{w^2}{(x-1)^2}=(x-w)^2$, with $w=-3$

$\dfrac{9}{(x-1)^2}=(x+3)^2 \Rightarrow (x+3)^2(x-1)^2=9$

$\Rightarrow (x+3)(x-1)=\pm 3$

Case 1

$\Rightarrow (x+3)(x-1)=3$

$\Rightarrow (x^2+2x-3)=3$

$x^2+2x-6=0$

$x=-1\pm\sqrt{7}$

Case 2

$\Rightarrow (x+3)(x-1)=-3$

$\Rightarrow (x^2+2x-3)=-3$

$x^2+2x=0$

$x=0,\ x=-2$

so $x=0,\ x=-2,\ x=-1\pm\sqrt{7}$

This question was not answered well. Many students expanded the expression and formed a quartic equation but did not proceed further. Students who took the more efficient approach of taking square roots to form two quadratic equations generally reached the correct solutions. Many students did not consider ± 3 in their solution process and thus simplified the solution process inappropriately. Some students used incorrect null factor law interpretations.

Question 9b.i

Marks	0	1	2	Average
%	43	34	22	0.8

$y=(x-1)(x-w)$

Use either symmetry, derivative = 0 or $x=\frac{-b}{2a}$

$$x_{\text{tp}}=\frac{1+w}{2}$$

$$y_{\text{tp}}=\left(\frac{1+w}{2}-1\right)\left(\frac{1+w}{2}-w\right)$$

$$=\left(\frac{1+w-2}{2}\right)\left(\frac{1+w-2w}{2}\right)$$

$$=\left(\frac{w-1}{2}\right)\left(\frac{1-w}{2}\right)$$

$$=-\frac{1}{4}\left(w-1\right)^2=-\frac{1}{4}(w^2-2w+1)$$

$$\therefore \text{ the minimum t.p is} =\begin{cases}\left(\frac{1+w}{2},-\frac{1}{4}\left(w-1\right)^2\right)\\ \left(\frac{1+w}{2},\frac{1}{4}\left(w-1\right)(1-w)\right)\\ \left(\frac{1+w}{2},\frac{-1+2w-w^2}{4}\right)\end{cases}$$

Any equivalent form.

There were many ways the answer could be expressed in this question and students were not required to give their answer in a particular form. Generally, students who used symmetry were able to find the correct x-coordinate. Many students made mistakes when combining fractions to find the y-coordinate. Common errors included writing the coordinates as $\left(\frac{w+1}{2},-\frac{\left(w-1^2\right)}{2}\right)$, $\left(\frac{1+w}{2},\frac{-1+2w-w^2}{2}\right)$ or $\left(\frac{1+w}{2},-\frac{w^2-1}{4}\right)$.

Question 9b.ii

Marks	0	1	2	Average
%	94	3	4	0.1

Method 1: DOPS consideration

Consider again $f(x)=g(x)$ with parameter $w>0$.

$$\frac{w^2}{(x-1)^2}=(x-w)^2$$

$$\Rightarrow (x-w)^2(x-1)^2=w^2$$

$$\Rightarrow \left[(x-w)(x-1)\right]^2-w^2=0$$

$$\Rightarrow \left[\underbrace{(x-w)(x-1)}_{\text{graph in b.i.}}-w\right]\left[\underbrace{(x-w)(x-1)}_{\text{graph in b.i.}}+w\right]=0$$

This can have three solutions if one factor yields one solution and the other yields two.

Case 1

If the first factor has only one solution:

$(x-w)(x-1)=w$, there is only one point of intersection so the minimum must touch $y=w$.

$$-\frac{1}{4}(w-1)^2=w$$

$$\frac{1}{4}(w-1)^2+w=0$$

$$w^2-2w+1+4w=0$$

$$w^2+2w+1=0$$

$$(w+1)^2=0$$

$$w=-1 \quad \text{rejected as } w>0$$

Case 2

If the second factor has only one solution:

$$(x-w)(x-1)=-w$$
$$-\frac{1}{4}(w-1)^2=-w$$
$$\frac{1}{4}(w-1)^2-w=0$$
$$(w-1)^2-4w=0$$
$$w^2-6w+1=0$$
$$w^2-6w+9=8$$
$$(w-3)^2=8$$
$$w-3=\pm 2\sqrt{2}$$
$$\therefore \quad w=3\pm 2\sqrt{2} \quad (\text{both positive, OK})$$

Method 2: ± case with minimum consideration

Consider again $f(x)=g(x)$ with parameter $w>0$.

$$\frac{w^2}{(x-1)^2}=(x-w)^2$$
$$\Rightarrow (x-w)^2(x-1)^2=w^2$$
$$\Rightarrow (x-w)(x-1)=\pm w$$

For this equation to have 3 solutions we must have the minimum turning point at $y=-w$

$$-\frac{1}{4}(w-1)^2=-w$$
$$\frac{1}{4}(w-1)^2-w=0$$
$$(w-1)^2-4w=0$$
$$w^2-6w+1=0$$
$$w^2-6w+9=8$$
$$(w-3)^2=8$$
$$w-3=\pm 2\sqrt{2}$$
$$\therefore \quad w=3\pm 2\sqrt{2} \quad (\text{both positive, OK})$$

Method 3: ± case with discriminant

Consider again $f(x) = g(x)$ with parameter $w > 0$.

$$\frac{w^2}{(x-1)^2} = (x-w)^2$$

$$\Rightarrow (x-w)^2(x-1)^2 = w^2$$
$$\Rightarrow (x-w)(x-1) = \pm w$$

Case 1

$$\begin{aligned} (x-w)(x-1) &= w \\ \Rightarrow x^2 - (w+1)x + w &= w \\ x(x-(w+1)) &= 0 \\ x = 0,\ x &= w+1 \end{aligned}$$

This gives two solutions, as $w > 0$.

Case 2: need this to give only one solution for there to be 3 solutions in total.

$$\begin{aligned} (x-w)(x-1) &= -w \\ \Rightarrow x^2 - (w+1)x + w &= -w \\ x^2 - (w+1)x + 2w &= 0 \end{aligned}$$

need $\Delta = 0$ for one solution

$$\begin{aligned} \Delta &= (w+1)^2 - 4 \times 1 \times 2w \\ \Delta &= w^2 + 2w + 1 - 8w \\ \Delta &= w^2 - 6w + 1 = 0 \\ w &= \frac{6 \pm \sqrt{36-4}}{2} \\ &= \left\{ \begin{array}{l} \dfrac{6 \pm \sqrt{32}}{2} \\ \dfrac{6 \pm 4\sqrt{2}}{2} \\ 3 \pm 2\sqrt{2} \end{array} \right. \end{aligned}$$

Method 4: solving quartic equal to a value

$$\frac{w^2}{(x-1)^2}=(x-w)^2$$

$$\Rightarrow (x-w)^2(x-1)^2=w^2$$

For this equation to have three solutions, $(x-w)^2(x-1)^2$ must have a local maximum with y-value equal to w^2.

Local maximum occurs halfway between intercepts at

$x_{\text{tp}}=\frac{1+w}{2}$ same value as in **part b.i**.

So $y_{\text{tp}}=\left(-\frac{1}{4}(w-1)^2\right)^2$

Now solve

$$\left(-\frac{1}{4}(w-1)^2\right)^2=w^2$$

$$-\frac{1}{4}(w-1)^2=\pm w$$

And then either consider cases as above to get

$$w=\begin{cases}\frac{6\pm\sqrt{32}}{2}\\ \frac{6\pm 4\sqrt{2}}{2}\\ 3\pm 2\sqrt{2}\end{cases}$$

Method 5: quartic expansion and solve

Expand and then use polynomial factorisation to solve (including quadratic formula)

$$\frac{(w-1)^4}{16} = w^2$$

$$w^4 - 4w^3 + 6w^2 - 4w + 1 = 16w^2$$

$$w^4 - 4w^3 - 10w^2 - 4w + 1 = 0$$

$$(w+1)(w^3 - 5w^2 - 5w + 1) = 0$$

$$(w+1)^2(w^2 - 6w + 1) = 0$$

$$\text{since } w > 0,\ w = \frac{6 \pm \sqrt{36-4}}{2}$$

$$w = \begin{cases} \dfrac{6 \pm \sqrt{32}}{2} \\ \dfrac{6 \pm 4\sqrt{2}}{2} \\ 3 \pm 2\sqrt{2} \end{cases}$$

There were many ways this question could be approached, however very few students made significant progress. Those students who approached the solution by forming $(x-w)(x-1) = \pm w$ and then using a discriminant approach (method 3 as described above), usually progressed further towards a complete solution. Some students were able to obtain an expanded quartic equation but then did not proceed any further. Few students used a *hence* approach and thus did not identify a connection between the turning point and the number of solutions. Those students who were able to demonstrate clear mathematical communication skills (listing all the possible solutions, compare them, discuss the nature of turning points or discuss the impact of the discriminant on the number of solutions) were the most successful.